Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 56 (2006) 2573–2591
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Some extensions of the Einstein–Dirac equation

Eui Chul Kim *

Department of Mathematics, College of Education, Andong National University, Andong 760-749, South Korea

Received 7 April 2005; accepted 28 January 2006
Available online 10 March 2006

Abstract

We considered an extension of the standard functional for the Einstein–Dirac equation where the Dirac operator is replaced by the square of the Dirac operator and a real parameter controlling the length of spinors is introduced. For one distinguished value of the parameter, the resulting Euler–Lagrange equations provide a new type of Einstein–Dirac coupling. We establish a special method for constructing global smooth solutions of a newly derived Einstein–Dirac system called the *CL-Einstein–Dirac equation of type II* (see Definition 3.1).

MSC: 53C25; 53C27; 83C05

Keywords: Riemannian spin manifold; Einstein–Dirac equation; Calculus of variations

1. Introduction

Let $(Q^{n,r}, \eta)$ be an n-dimensional (connected smooth) pseudo-Riemannian manifold, where the index r is the number of negative eigenvalues of the metric η. Assume that $(Q^{n,r}, \eta)$ is space- and time-oriented and has a fixed spin structure [1]. For simplicity, we will often write Q to mean $Q^{n,r}$. Let $\Sigma(Q) = \Sigma(Q)_\eta$ denote the spinor bundle of $(Q^{n,r}, \eta)$ equipped with the $\mathrm{Spin}^+(n, r)$-equivariant nondegenerate complex product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\eta$, and let $(\cdot,\cdot) = \mathrm{Re}\langle\cdot,\cdot\rangle$ denote the real part of $\langle\cdot,\cdot\rangle$. Let $\mathrm{Ric} = \mathrm{Ric}_\eta$ and $S = S_\eta$ be the Ricci tensor and the scalar curvature of $(Q^{n,r}, \eta)$, respectively. Let $D = D_\eta$ be the Dirac operator acting on sections $\psi \in \Gamma(\Sigma(Q))$ of the spinor bundle $\Sigma(Q)$. Then the standard functional for the Einstein–Dirac equation is given

* Tel.: +82 54 820 5544; fax: +82 54 820 6041.
E-mail address: eckim@andong.ac.kr.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.01.008

by

$$W_1(\eta, \psi) = \int \left\{ aS_\eta + b + \epsilon \nu_1(\psi, \psi) - \epsilon((\sqrt{-1})^r D_\eta \psi, \psi) \right\} \mu_\eta, \tag{1.1}$$

where $a, b, \epsilon, \nu_1 \in \mathbb{R}$, $\epsilon \neq 0$, are real numbers and μ_η is the volume form of $(Q^{n,r}, \eta)$. The Euler–Lagrange equations (called the Einstein–Dirac equation) are the Dirac equation

$$(\sqrt{-1})^r D\psi = \nu_1 \psi \tag{1.2}$$

and the Einstein equation

$$a \left\{ \mathrm{Ric} - \frac{S}{2}\eta \right\} - \frac{b}{2}\eta = \frac{\epsilon}{4} T_1 \tag{1.3}$$

coupled via a symmetric tensor field T_1,

$$T_1(X, Y) = \left((\sqrt{-1})^r \{ X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi \}, \psi \right), \tag{1.4}$$

where X, Y are vector fields on $Q^{n,r}$ and the dot "·" indicates the Clifford multiplication. Observe that the system (1.2)–(1.4) contains four differential operators, namely, the spin connection ∇, the Dirac operator D, the Ricci tensor Ric and the scalar curvature S. The spin connection and the Dirac operator act on spinor fields and are operators of first-order, while the Ricci tensor and the scalar curvature are second-order operators acting on metrics. Therefore, it is natural to ask whether one can derive such Euler–Lagrange equations from the functional

$$W_2(\eta, \psi) = \int \left\{ aS_\eta + b + \epsilon \nu_2(\psi, \psi) - \epsilon((D_\eta \circ D_\eta)(\psi), \psi) \right\} \mu_\eta, \qquad \nu_2 \in \mathbb{R}, \tag{1.5}$$

that generalize the system (1.2)–(1.4) and all the involved operators acting on spinor fields are of second-order. In Section 2 we will show that the answer to the question is positive and (1.5) yields in fact the following system (see Theorem 2.1):

$$D^2 \psi = \nu_2 \psi, \qquad a \left\{ \mathrm{Ric} - \frac{S}{2}\eta \right\} - \frac{b}{2}\eta = \frac{\epsilon}{4} T_2, \tag{1.6}$$

where T_2 is a symmetric tensor field defined by

$$\begin{aligned} T_2(X, Y) = {} & (X \cdot \nabla_Y (D\psi) + Y \cdot \nabla_X (D\psi), \psi) \\ & + (-1)^r \, (X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, D\psi) \, . \end{aligned} \tag{1.7}$$

In this paper the system (1.2)–(1.4) is called the *classical Einstein–Dirac equation of type I* [5–7] and the system (1.6) and (1.7) the *classical Einstein–Dirac equation of type II*.

Let us turn to another situation where a real parameter controlling the length of spinors is introduced. Let $\varphi = \varphi_\eta$ be a spinor field on $(Q^{n,r}, \eta)$ such that either $(\varphi, \varphi) > 0$ at all points or $(\varphi, \varphi) < 0$ at all points. Fix a shorthand notation

$$\varphi^k := (\sigma\varphi, \varphi)^k \varphi, \quad \varphi^0 := \varphi,$$

where $k \in \mathbb{R}$ is a real number and $\sigma = \sigma_\varphi \in \mathbb{R}$ is a constant defined by

$$\sigma = 1 \quad \text{if } (\varphi, \varphi) > 0 \quad \text{and} \quad \sigma = -1 \quad \text{if } (\varphi, \varphi) < 0.$$

Combining the functional (1.1) with (1.5), we extend the spinorial part as

$$W(\eta, \varphi) = \int \left\{ a S_\eta + b + \epsilon \nu (\sigma \varphi^k, \varphi^k) - \epsilon (\sigma P_\eta(\varphi^k), \varphi^k) \right\} \mu_\eta, \qquad \nu \in \mathbb{R}, \tag{1.8}$$

where $P_\eta = (\sqrt{-1})^r D_\eta$ or $P_\eta = D_\eta \circ D_\eta$, and look at the Euler–Lagrange equations derived from (1.8). We will show in Section 3 (see Theorem 3.1) that, when $k \neq -\frac{1}{2}$, the Euler–Lagrange equations of (1.8) are actually equivalent to the system (1.2)–(1.4) or to the system (1.6) and (1.7) depending on a choice of P_η. However, in the distinguished case $k = -\frac{1}{2}$ in which the length $|\varphi^k| = \pm 1$ becomes constant, we are led to a new Einstein–Dirac system, i.e.,

$$P_\eta \psi = f \psi, \qquad a \left\{ \mathrm{Ric} - \frac{S}{2} \eta \right\} - \frac{c}{2} \eta = \frac{\epsilon}{4} T - \frac{\epsilon}{2} f \eta, \qquad a, c, \epsilon \in \mathbb{R}, \tag{1.9}$$

where ψ is of constant length $|\psi| = \pm 1$ and $f : Q^{n,r} \longrightarrow \mathbb{R}$ is a real-valued function and T is a symmetric tensor field defined by

$$T(X, Y) = \left(\sigma (\sqrt{-1})^r \{ X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi \}, \psi \right) \tag{1.10}$$

if $P_\eta = (\sqrt{-1})^r D_\eta$ and by

$$\begin{aligned} T(X, Y) = {} & \sigma \left(X \cdot \nabla_Y (D\psi) + Y \cdot \nabla_X (D\psi), \psi \right) \\ & + \sigma (-1)^r \left(X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, D\psi \right) \end{aligned} \tag{1.11}$$

if $P_\eta = D_\eta \circ D_\eta$, respectively. The system (1.9)–(1.11) will be called the *CL-Einstein–Dirac equation of type I* if $P_\eta = (\sqrt{-1})^r D_\eta$ and the *CL-Einstein–Dirac equation of type II* if $P_\eta = D_\eta \circ D_\eta$, respectively ("CL" means the "constant length" of spinors). A non-trivial spinor field ψ on $(Q^{n,r}, \eta)$ is called a *CL-Einstein spinor* of type I (resp. type II) if it satisfies the CL-Einstein–Dirac equation of type I (resp. type II). It will be pointed out (see Remark 3.1) why one cannot weaken the "constant length" condition for CL-Einstein spinors.

Sections 4 and 5 of the paper are devoted to establishing a special method for constructing global (smooth) solutions of the CL-Einstein–Dirac equation of type II. The essential idea of this construction is the fact that, under conformal change of metrics, the CL-Einstein–Dirac equation of type II behave in a relatively stable way (more stable than the CL-Einstein–Dirac equation of type I and both types of classical Einstein–Dirac equation). More precisely, we show in Section 4 that if $(Q^{n,r}, \eta)$ admits a non-trivial spinor field ψ, called a *reduced weakly parallel spinor*, satisfying the differential equation in Definition 4.3, then over the manifold $(Q^{n,r}, \overline{\eta} = \mathrm{e}^u \eta)$ with conformally changed metric $\overline{\eta} = \mathrm{e}^u \eta$ the pullback $\overline{\psi}$ of ψ becomes a CL-Einstein spinor of type II (see Theorem 4.2). Parallel spinors [8] are trivial examples for reduced weakly parallel spinors. In Section 5 we will provide examples for reduced weakly parallel spinors that are not parallel spinors (see Theorem 5.2).

2. Coupling of the square of the Dirac operator to the Einstein equation

We first recall the process of obtaining the classical Einstein–Dirac equation of type I in pseudo-Riemannian signature [6,7]. Applying the process to the behaviour of the square of the Dirac operator under change of metrics, we then derive the classical Einstein–Dirac equation of type II.

Let h be a symmetric (0, 2)-tensor field on $(Q^{n,r}, \eta)$, and let H be the (1, 1)-tensor field induced by h via $h(X, Y) = \eta(X, H(Y))$. Then the tensor field $\overline{\eta}$ defined by

$$\overline{\eta}(X, Y) = \eta(X, \mathrm{e}^{H}(Y)) = \eta(\mathrm{e}^{\frac{H}{2}}(X), \mathrm{e}^{\frac{H}{2}}(Y)) \tag{2.1}$$

is a pseudo-Riemannian metric of the same index r. Let $K := \mathrm{e}^{\frac{H}{2}}$ and let Λ be the (1, 2)-tensor field defined by

$$\begin{aligned} 2\eta(\Lambda(X, Y), Z) = {}& \eta\left(Z, K\{(\nabla^{\eta}_{K^{-1}(X)} K^{-1})(Y)\} - K\{(\nabla^{\eta}_{K^{-1}(Y)} K^{-1})(X)\}\right) \\ & + \eta\left(Y, K\{(\nabla^{\eta}_{K^{-1}(Z)} K^{-1})(X)\} - K\{(\nabla^{\eta}_{K^{-1}(X)} K^{-1})(Z)\}\right) \\ & + \eta\left(X, K\{(\nabla^{\eta}_{K^{-1}(Z)} K^{-1})(Y)\} - K\{(\nabla^{\eta}_{K^{-1}(Y)} K^{-1})(Z)\}\right). \end{aligned}$$

Then the Levi-Civita connections $\nabla^{\overline{\eta}}$ and ∇^{η} are related by

$$\nabla^{\overline{\eta}}_{K^{-1}(X)}\left(K^{-1}(Y)\right) = K^{-1}\left(\nabla^{\eta}_{K^{-1}(X)} Y\right) + K^{-1}\{\Lambda(X, Y)\}. \tag{2.2}$$

Let $\widehat{K} : \Sigma(Q)_{\overline{\eta}} \longrightarrow \Sigma(Q)_{\eta}$ be a natural isomorphism preserving the inner product of spinors and the Clifford multiplication with

$$\langle \widehat{K}(\varphi), \widehat{K}(\psi)\rangle_{\eta} = \langle \varphi, \psi\rangle_{\overline{\eta}}, \qquad (KX)\cdot(\widehat{K}\psi) = \widehat{K}(X\cdot\psi) \tag{2.3}$$

for all $X \in \Gamma(T(Q))$, $\varphi, \psi \in \Gamma(\Sigma(Q)_{\overline{\eta}})$, where the dot "·" in the latter relation indicates the Clifford multiplication with respect to η and $\overline{\eta}$, respectively. Let $(E_1, \ldots, E_n)$ be a local η-orthonormal frame field on $(Q^{n,r}, \eta)$. For brevity we introduce the notation $\chi(i) := \eta(E_i, E_i)$ and $\chi(i_1 \ldots i_s) := \chi(i_1)\chi(i_2)\cdots\chi(i_s)$ for $1 \le s \le n$. Then, because of (2.2), the spinor derivatives ∇^{η}, $\nabla^{\overline{\eta}}$ are related by [4]

$$\left\{\widehat{K} \circ \nabla^{\overline{\eta}}_{K^{-1}(E_j)} \circ \left(\widehat{K}\right)^{-1}\right\}(\psi) = \nabla^{\eta}_{K^{-1}(E_j)}\psi + \frac{1}{4}\sum_{k,l=1}^{n} \chi(kl)\Lambda_{jkl} E_k \cdot E_l \cdot \psi, \tag{2.4}$$

where $\Lambda_{jkl} := \eta(\Lambda(E_j, E_k), E_l)$, and the Dirac operators D_{η}, $D_{\overline{\eta}}$ by

$$\begin{aligned} &\left\{\widehat{K} \circ D_{\overline{\eta}} \circ \left(\widehat{K}\right)^{-1}\right\}(\psi) \\ &= \sum_{i=1}^{n} \chi(i) E_i \cdot \nabla^{\eta}_{K^{-1}(E_i)}\psi + \frac{1}{4}\sum_{j,k,l=1}^{n} \chi(jkl)\Lambda_{jkl} E_j \cdot E_k \cdot E_l \cdot \psi \\ &= \sum_{i=1}^{n} \chi(i) E_i \cdot \nabla^{\eta}_{K^{-1}(E_i)}\psi - \frac{1}{2}\sum_{j,k=1}^{n} \chi(jk)\Lambda_{jjk} E_k \cdot \psi \\ &\quad + \frac{1}{2}\sum_{j<k<l}^{n} \chi(jkl)(\Lambda_{jkl} + \Lambda_{klj} + \Lambda_{ljk}) E_j \cdot E_k \cdot E_l \cdot \psi. \end{aligned} \tag{2.5}$$

In order to compute the infinitesimal variation of the Dirac operator, we consider a one-parameter family of metrics of index r,

$$\eta_t(X, Y) := \eta(X, \mathrm{e}^{tH}(Y)) = \eta(\mathrm{e}^{\frac{tH}{2}}(X), \mathrm{e}^{\frac{tH}{2}}(Y)), \qquad \eta_o := \eta, t \in \mathbb{R}, \tag{2.6}$$

which is generated by a symmetric (0, 2)-tensor field h on $(Q^{n,r}, \eta)$. Let Λ_t be the (1, 2)-tensor in (2.2) determined by the pair $(\nabla^{\eta_t}, \nabla^{\eta})$ of the Levi-Civita connections (with $K_t = e^{\frac{tH}{2}}$). Let Ω_t be a 3-form generated by the tensor Λ_t via

$$\Omega_t(X, Y, Z) = \eta(\Lambda_t(X, Y), Z) + \eta(\Lambda_t(Y, Z), X) + \eta(\Lambda_t(Z, X), Y). \tag{2.7}$$

Then direct computations show:

Lemma 2.1.

$$\left.\frac{d}{dt}\right|_{t=0} \{\Lambda_t(X, Y) - \Lambda_t(Y, X)\} = -\frac{1}{2}(\nabla^{\eta}_X H)(Y) + \frac{1}{2}(\nabla^{\eta}_Y H)(X),$$
$$\left.\frac{d}{dt}\right|_{t=0} \eta(\Lambda_t(X, Y), Z) = \frac{1}{2}\eta((\nabla^{\eta}_Y H)(X), Z) - \frac{1}{2}\eta((\nabla^{\eta}_Z H)(X), Y),$$
$$\left.\frac{d}{dt}\right|_{t=0} \Omega_t(X, Y, Z) = 0.$$

Applying Lemma 2.1 to (2.5), we arrive at the variation formula of the Dirac operator:

$$\left.\frac{d}{dt}\right|_{t=0} \left\{\widehat{K_t} \circ D_{\eta_t} \circ \left(\widehat{K_t}\right)^{-1}\right\}(\psi)$$
$$= -\frac{1}{2}\sum_{j=1}^{n} \chi(j) h(E_j) \cdot \nabla^{\eta}_{E_j}\psi - \frac{1}{4}\mathrm{div}_{\eta}(h) \cdot \psi + \frac{1}{4}\mathrm{grad}_{\eta}(\mathrm{Tr}_{\eta}(h)) \cdot \psi. \tag{2.8}$$

Recall [1] that for the standard complex product $\langle \cdot, \cdot \rangle$ on the spinor bundle $\Sigma(Q)$, the relation

$$\langle X \cdot \varphi, \psi \rangle + (-1)^r \langle \varphi, X \cdot \psi \rangle = 0 \tag{2.9}$$

holds for all vector fields X and for all spinor fields φ, ψ. Taking the real part of (2.9) gives some simple but crucial identities:

$$((\sqrt{-1})^r X \cdot \psi, \psi) = 0, \tag{2.10}$$
$$(X \cdot \psi, Y \cdot \psi) = (-1)^r \eta(X, Y)(\psi, \psi), \tag{2.11}$$
$$(X \cdot Y \cdot \psi, \psi) = -\eta(X, Y)(\psi, \psi). \tag{2.12}$$

Let Sym(0, 2) denote the space of all symmetric (0, 2)-tensor fields on $(Q^{n,r}, \eta)$, and let $((\cdot, \cdot)) = ((\cdot, \cdot))_{\eta}$ denote the naturally induced metric on the space Sym(0, 2). Denote by $\psi_{\eta_t} = (\widehat{K_t})^{-1}(\psi) \in \Gamma(\Sigma(Q)_{\eta_t})$ the pullback of $\psi = \psi_{\eta} \in \Gamma(\Sigma(Q)_{\eta})$ via natural isomorphism $\widehat{K_t}$ (see (2.3)). Then (2.8) and (2.10) together give the formula (1.4) for the first type energy–momentum tensor T_1:

$$\left.\frac{d}{dt}\right|_{t=0} \left((\sqrt{-1})^r D_{\eta_t}\psi_{\eta_t}, \psi_{\eta_t}\right) = -\frac{1}{4}((T_1, h)), \tag{2.13}$$

where

$$T_1(X, Y) = \left((\sqrt{-1})^r \{X \cdot \nabla^{\eta}_Y \psi + Y \cdot \nabla^{\eta}_X \psi\}, \psi\right). \tag{2.14}$$

Moreover, using (2.8) and (2.9) and noting that $(\sqrt{-1})^r D_{\eta}$ is symmetric with respect to the L^2-product, we can derive the formula (1.7) for the second type energy–momentum tensor T_2.

Lemma 2.2. *Let U be an open subset of $Q^{n,r}$ with compact closure, and let h be a symmetric tensor field with support in U. Then for any spinor field ψ on $(Q^{n,r}, \eta)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_U \left((D_{\eta_t} \circ D_{\eta_t})(\psi_{\eta_t}), \psi_{\eta_t}\right) \mu_\eta = -\frac{1}{4}\int_U ((T_2, h))\mu_\eta,$$

where

$$T_2(X, Y) = \left(X \cdot \nabla^\eta_Y (D_\eta \psi) + Y \cdot \nabla^\eta_X (D_\eta\psi), \psi\right) + (-1)^r \left(X \cdot \nabla^\eta_Y \psi + Y \cdot \nabla^\eta_X \psi, D_\eta\psi\right). \tag{2.15}$$

Proof. Letting $D = D_\eta$ and $\psi = \psi_\eta$, we compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_U \left((D_{\eta_t} \circ D_{\eta_t})(\psi_{\eta_t}), \psi_{\eta_t}\right)_{\eta_t} \mu_\eta \\
&= \int_U \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\widehat{K}_t D_{\eta_t})(D\psi)_{\eta_t}, \psi\right)\mu_\eta + \int_U \left(D_\eta\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\widehat{K}_t D_{\eta_t})(\psi_{\eta_t})\right), \psi\right)\mu_\eta \\
&= \int_U \left(-\frac{1}{2}\sum_{j=1}^n \chi(j) h(E_j)\cdot \nabla^\eta_{E_j}(D\psi) - \frac{1}{4}\mathrm{div}_\eta(h)\cdot (D\psi)\right. \\
&\quad \left. + \frac{1}{4}\mathrm{grad}_\eta(\mathrm{Tr}_\eta(h))\cdot(D\psi), \psi\right)\mu_\eta \\
&\quad + \int_U \left((\sqrt{-1})^{3r}\left\{-\frac{1}{2}\sum_{j=1}^n \chi(j)h(E_j)\cdot\nabla^\eta_{E_j}\psi - \frac{1}{4}\mathrm{div}_\eta(h)\cdot\psi\right.\right. \\
&\quad \left.\left. + \frac{1}{4}\mathrm{grad}_\eta(\mathrm{Tr}_\eta(h))\cdot\psi\right\}, (\sqrt{-1})^r D_\eta\psi\right)\mu_\eta \\
&= -\frac{1}{2}\int_U\left(\sum_{i=1}^n \chi(i)h(E_i)\cdot\nabla^\eta_{E_i}(D\psi), \psi\right)\mu_\eta - \frac{(-1)^r}{2} \\
&\quad \times \int_U \left(\sum_{i=1}^n \chi(i)h(E_i)\cdot\nabla^\eta_{E_i}\psi, D\psi\right)\mu_\eta \\
&= -\frac{1}{4}\int_U ((T_2, h))\mu_\eta. \quad \square
\end{aligned}$$

We further need to recall the well-known formulas for the variation of the volume form and the scalar curvature, which one easily obtains from (2.6) and from the pseudo-Riemannian version of the second formula in Proposition 2.2 of [7].

Lemma 2.3 (*See [3]*). *Let U be an open subset of $Q^{n,r}$ with compact closure, and let h be a symmetric tensor field with support in U. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mu_{\eta_t} = \frac{1}{2}((\eta, h))\mu_\eta,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_U S_{\eta_t}\mu_\eta = -\int_U ((\mathrm{Ric}_\eta, h))\mu_\eta.$$

Making use of Lemmas 2.2 and 2.3 and following the proof of Theorem 2.1 of [6], we now establish the main result of this section.

Theorem 2.1. *Let $Q^{n,r}$ be a pseudo-Riemannian spin manifold. Fix the notation P_η to mean either $P_\eta = (\sqrt{-1})^r D_\eta$ or $P_\eta = D_\eta \circ D_\eta$. Then, a pair (η_o, ψ_o) is a critical point of the Lagrange functional*

$$W(\eta, \psi) = \int_U \left\{aS_\eta + b + \epsilon\nu(\psi_\eta, \psi_\eta)_\eta - \epsilon(P_\eta(\psi), \psi)_\eta\right\}\mu_\eta, \quad a, b, \epsilon, \nu \in \mathbb{R}, \epsilon \neq 0,$$

for all open subsets U of $Q^{n,r}$ with compact closure if and only if (η_o, ψ_o) is a solution of the following system of differential equations:

$$P_\eta(\psi) = \nu\psi \qquad \textit{and} \qquad a\left\{\mathrm{Ric}_\eta - \frac{1}{2}S_\eta\eta\right\} - \frac{b}{2}\eta = \frac{\epsilon}{4}T, \tag{2.16}$$

where T is a symmetric tensor field defined by (2.14) or by (2.15) depending on a choice of P_η.

We close the section with generalizing Definition 2.1 and 3.1 of [6].

Definition 2.1. (i) A non-trivial spinor field ψ on $(Q^{n,r}, \eta)$, $n \geq 3$, is called an *Einstein spinor of type I* for the eigenvalue $(\sqrt{-1})^{3r}\nu_1$, $\nu_1 \in \mathbb{R}$, if it is a solution of the system (1.2)–(1.4).

(ii) A non-trivial spinor field ψ on $(Q^{n,r}, \eta)$, $n \geq 3$, is called an *Einstein spinor of type II* for the eigenvalue $\nu_2 \in \mathbb{R}$ if it is a solution of the system (1.6) and (1.7).

Definition 2.2. Assume that $a(n-2)S + bn(a, b \in \mathbb{R})$ does not vanish at any point of $(Q^{n,r}, \eta)$, $n \geq 3$. A non-trivial spinor field ψ on $(Q^{n,r}, \eta)$ is called a *weak Killing spinor* (briefly, WK-spinor) with WK-number $(\sqrt{-1})^{3r}\nu_1 \neq 0$, $\nu_1 \in \mathbb{R}$, if ψ is a solution of the differential equation

$$\nabla_X\psi = (\sqrt{-1})^{3r}\beta(X)\cdot\psi + n\alpha(X)\psi + X\cdot\alpha\cdot\psi, \tag{2.17}$$

where α is a 1-form and β is a symmetric tensor field defined by

$$\alpha = \frac{a(n-2)\mathrm{d}S}{2(n-1)\{a(n-2)S + bn\}}$$

and

$$\beta = \frac{2\nu_1}{a(n-2)S + bn}\left\langle a\left\{\mathrm{Ric} - \frac{1}{2}S\eta\right\} - \frac{b}{2}\eta\right\rangle,$$

respectively.

Remark 2.1. As in the Riemannian case (see Theorem 3.1 of [6]), any pseudo-Riemannian WK-spinor ψ with positive length $(\psi, \psi) > 0$ (resp. negative length $(\psi, \psi) < 0$) becomes an Einstein spinor of type I: Since

$$\mathrm{d}\left(\frac{(\psi, \psi)}{a(n-2)S + bn}\right) = 0,$$

it follows that

$$\frac{(\psi, \psi)}{a(n-2)S + bn}$$

is constant on $Q^{n,r}$. One verifies easily that Eqs. (1.2)–(1.4) are indeed satisfied with

$$\epsilon = -\frac{a(n-2)S + bn}{\nu_1(\psi, \psi)}.$$

Remark 2.2. Evidently, the solution space of the type I classical Einstein–Dirac equation is a subspace of that of the type II classical Einstein–Dirac equation. Hence it is of interest to find such Einstein spinors of type II that are not Einstein spinors of type I. Let $(Q^{n,r}, \eta)$ admit a spinor field ψ satisfying the differential equation [2]

$$\nabla_X \psi = -(\sqrt{-1})^{3r+1} \frac{\nu_1}{n} X \cdot \psi.$$

Then the metric η is necessarily Einstein with scalar curvature

$$S = (-1)^{r+1} \frac{4(n-1)\nu_1^2}{n}.$$

If we choose the parameters a and b so as to be related by

$$b = -\frac{a(n-2)}{n} S = (-1)^r \frac{4a(n-1)(n-2)\nu_1^2}{n^2},$$

then ψ satisfies (1.6) and (1.7) with

$$\nu_2 = (-1)^{r+1}\nu_1^2 \qquad \text{and} \qquad a\left\{\mathrm{Ric} - \frac{S}{2}\eta\right\} - \frac{b}{2}\eta = \frac{\epsilon}{4} T_2 = 0.$$

However, ψ does not satisfy (1.2)–(1.4) in general.

3. Derivation of the CL-Einstein–Dirac equations

Let $\varphi = \varphi_\eta$ be a spinor field on $(Q^{n,r}, \eta)$ such that either $(\varphi, \varphi) > 0$ at all points or $(\varphi, \varphi) < 0$ at all points. We use the simplifying notation

$$\varphi^k := (\sigma\varphi, \varphi)^k \varphi, \qquad k \in \mathbb{R},$$

where $\sigma = \sigma_\varphi \in \mathbb{R}$ is a constant defined by

$$\sigma = 1 \quad \text{if } (\varphi, \varphi) > 0 \qquad \text{and} \qquad \sigma = -1 \quad \text{if } (\varphi, \varphi) < 0.$$

Via direct computations, one verifies easily the following variation formulas.

Lemma 3.1. *Let U be an open subset of $(Q^{n,r}, \eta)$ with compact closure, and let φ_c be a spinor field with support in U. Then we have*

(i)

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma(\varphi + t\varphi_c)^k, (\varphi + t\varphi_c)^k) = 2(2k+1)(\sigma\varphi, \varphi)^{2k}(\sigma\varphi, \varphi_c),$$

(ii)

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_U \left(\sigma P_\eta(\varphi + t\varphi_c)^k, (\varphi + t\varphi_c)^k\right) \mu_\eta$$
$$= 4k \int_U \left(\sigma P_\eta\{(\sigma\varphi, \varphi)^k \varphi\}, (\sigma\varphi, \varphi)^{k-1}\varphi\right) (\sigma\varphi, \varphi_c)\mu_\eta$$
$$+ 2 \int_U \left(\sigma P_\eta\{(\sigma\varphi, \varphi)^k \varphi\}, (\sigma\varphi, \varphi)^k \varphi_c\right) \mu_\eta,$$

where $P_\eta = (\sqrt{-1})^r D_\eta$ *or* $P_\eta = D_\eta \circ D_\eta$.

Theorem 3.1. *Let* $Q^{n,r}$ *be a pseudo-Riemannian spin manifold. Consider the Lagrange functional*

$$W(\eta, \varphi) = \int_U \left\{ a S_\eta + b + \epsilon\nu(\sigma\varphi^k, \varphi^k)_\eta - \epsilon(\sigma P_\eta(\varphi^k), \varphi^k)_\eta \right\} \mu_\eta$$

over open subsets U *of* $Q^{n,r}$ *with compact closure, where* $a, b, k, \epsilon, \nu \in \mathbb{R}, \epsilon \neq 0$, *are real numbers.*

(i) *In case of* $2k + 1 \neq 0$, *a pair* (η^*, φ^*) *is a critical point of* $W(\eta, \varphi)$ *for all open subsets* U *of* $Q^{n,r}$ *with compact closure if and only if* (η^*, φ^*) *is a solution of the following system of differential equations:*

$$P_\eta(\varphi^k) = \nu\varphi^k \qquad \text{and} \qquad a\left\{\mathrm{Ric}_\eta - \frac{1}{2}S_\eta\eta\right\} - \frac{b}{2}\eta = \frac{\epsilon}{4}T, \tag{3.1}$$

where T *is a symmetric tensor field defined by*

$$T(X, Y) = T_1(X, Y) = \left(\sigma(\sqrt{-1})^r\{X \cdot \nabla^\eta_Y \varphi^k + Y \cdot \nabla^\eta_X \varphi^k\}, \varphi^k\right) \tag{3.2}$$

if $P_\eta = (\sqrt{-1})^r D_\eta$ *and defined by*

$$T(X, Y) = T_2(X, Y) = \sigma\left(X \cdot \nabla^\eta_Y(D_\eta\varphi^k) + Y \cdot \nabla^\eta_X(D_\eta\varphi^k), \varphi^k\right)$$
$$+ \sigma(-1)^r\left(X \cdot \nabla^\eta_Y\varphi^k + Y \cdot \nabla^\eta_X\varphi^k, D_\eta\varphi^k\right) \tag{3.3}$$

if $P_\eta = D_\eta \circ D_\eta$, *respectively.*

(ii) *In the case of* $2k + 1 = 0$, *a pair* (η^*, φ^*) *is a critical point of* $W(\eta, \varphi)$ *for all open subsets* U *of* $Q^{n,r}$ *with compact closure if and only if* (η^*, φ^*) *is a solution of the following system of differential equations:*

$$P_\eta(\varphi^k) = f\varphi^k \tag{3.4}$$

and

$$a\left\{\mathrm{Ric}_\eta - \frac{1}{2}S_\eta\eta\right\} - \frac{b + \epsilon\nu}{2}\eta = \frac{\epsilon}{4}T - \frac{\epsilon}{2}f\eta, \tag{3.5}$$

where $f : Q^{n,r} \longrightarrow \mathbb{R}$ *is a real-valued function and* T *is a symmetric tensor field defined by* (3.2) *or by* (3.3) *depending on a choice of* P_η.

Proof. Let h be a symmetric tensor field with support in U, and let φ_c be a spinor field with support in U. Let η_t be a one-parameter family of metrics in (2.6). Using Lemma 3.1, we compute at $t = 0$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} W(\eta_t, \varphi + t\varphi_c) &= \frac{\mathrm{d}}{\mathrm{d}t} W(\eta_t, \varphi) + \frac{\mathrm{d}}{\mathrm{d}t} W(\eta, \varphi + t\varphi_c) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_U a S_{\eta_t} \mu_\eta + \frac{\mathrm{d}}{\mathrm{d}t} \int_U a S_\eta \mu_{\eta_t} + \frac{\mathrm{d}}{\mathrm{d}t} \int_U b \mu_{\eta_t} + \frac{\mathrm{d}}{\mathrm{d}t} \int_U \epsilon\nu(\sigma\varphi^k, \varphi^k)\mu_{\eta_t} \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t} \int_U \epsilon(\sigma P_\eta(\varphi^k), \varphi^k)\mu_{\eta_t} - \frac{\mathrm{d}}{\mathrm{d}t} \int_U \epsilon(\sigma P_{\eta_t}(\varphi^k_{\eta_t}), \varphi^k_{\eta_t})\mu_\eta \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t} \int_U \epsilon\nu(\sigma(\varphi + t\varphi_c)^k, (\varphi + t\varphi_c)^k)\mu_\eta - \frac{\mathrm{d}}{\mathrm{d}t} \int_U \epsilon(\sigma P_\eta(\varphi + t\varphi_c)^k, (\varphi + t\varphi_c)^k)\mu_\eta \\
&= \int_U \left(\left(-a\mathrm{Ric}_\eta + \frac{a}{2} S_\eta \eta + \frac{b}{2}\eta + \frac{\epsilon}{4} T + \frac{\epsilon\nu}{2}(\sigma\varphi^k, \varphi^k)\eta - \frac{\epsilon}{2}(\sigma P_\eta(\varphi^k), \varphi^k)\eta,\ h \right) \right) \mu_\eta \\
&\quad + \int_U \Big(2\epsilon\nu(2k+1)(\sigma\varphi, \varphi)^{2k} \cdot \sigma\varphi - 4\epsilon k(\sigma\varphi, \varphi)^{-1}(\sigma P_\eta(\varphi^k), \varphi^k) \cdot \sigma\varphi \\
&\quad - 2\epsilon(\sigma\varphi, \varphi)^k \cdot \sigma P_\eta(\varphi^k), \varphi_c \Big) \mu_\eta.
\end{aligned}$$

It follows that a pair (η^*, φ^*) is a critical point of the functional $W(\eta, \varphi)$ for all open subsets U of $Q^{n,r}$ with compact closure if and only if it is a solution of the equations

$$\frac{\epsilon}{4} T = a\mathrm{Ric}_\eta - \frac{a}{2} S_\eta \eta - \frac{b}{2}\eta - \frac{\epsilon\nu}{2}(\sigma\varphi^k, \varphi^k)\eta + \frac{\epsilon}{2}(\sigma P_\eta(\varphi^k), \varphi^k)\eta \tag{3.6}$$

and

$$P_\eta(\varphi^k) = -2k(\sigma\varphi, \varphi)^{-2k-1}(\sigma P_\eta(\varphi^k), \varphi^k)\varphi^k + \nu(2k+1)\varphi^k. \tag{3.7}$$

Inner product of (3.7) with $\sigma \cdot \varphi^k$ gives

$$0 = (2k+1)\left\{ (\sigma P_\eta(\varphi^k), \varphi^k) - \nu(\sigma\varphi^k, \varphi^k) \right\}, \tag{3.8}$$

and so, in the case of $2k + 1 \neq 0$, (3.6)–(3.8) imply part (i) of the theorem. Now we consider the other case $2k + 1 = 0$. In this case, $(\sigma\varphi^k, \varphi^k) = (\sigma\varphi, \varphi)^{2k+1} = 1$ and hence (3.7) gives

$$P_\eta(\varphi^k) = f\varphi^k \tag{3.9}$$

with $f := (\sigma P_\eta(\varphi^k), \varphi^k)$. Thus, (3.6) and (3.9) together prove part (ii) of the theorem. □

We observe that the system (3.1)–(3.3) is not new and is in fact equivalent to the classical system (2.16). We therefore focus our attention on the system (3.4) and (3.5) which is a new Einstein–Dirac system.

Definition 3.1. A non-trivial spinor field ψ on $(Q^{n,r}, \eta)$, $n \geq 3$, is called a *CL-Einstein spinor of type I* (resp. *type II*) *with characteristic function* f if it is of constant length $|\psi| = \pm 1$ and satisfies the system (1.9) and (1.10) (resp. (1.9) and (1.11)).

Remark 3.1. Let φ be a spinor field on $(Q^{n,r}, \eta)$ such that either $(\varphi, \varphi) > 0$ at all points or $(\varphi, \varphi) < 0$ at all points. Let T_1 and T_2 be symmetric tensor fields induced by φ as in (1.10) and

(1.11), respectively. Then, via direct computations, one finds that

$$\begin{aligned}\operatorname{div}(T_1)(X) &= \sigma \sum_{i=1}^{n} \chi(i)(\nabla_{E_i} T_1)(E_i, X)\\ &= \sigma\left((\sqrt{-1})^r \nabla_X (D\varphi), \varphi\right) - \sigma\left(\nabla_X \varphi, (\sqrt{-1})^r D\varphi\right) - \sigma\left((\sqrt{-1})^r X \cdot D^2\varphi, \varphi\right) \end{aligned}\tag{3.10}$$

and

$$\begin{aligned}\operatorname{div}(T_2)(X) &= \sigma\left(\nabla_X (D^2\varphi), \varphi\right) - \sigma\left(\nabla_X \varphi, D^2\varphi\right)\\ &\quad - \sigma\left(X \cdot D^3\varphi, \varphi\right) - (-1)^r \sigma\left(X \cdot D^2\varphi, D\varphi\right).\end{aligned}\tag{3.11}$$

(i) If $(\sqrt{-1})^r D\varphi = f_1\varphi$ for some function $f_1 : Q^{n,r} \longrightarrow \mathbb{R}$ and φ is of constant length $|\varphi| = \pm 1$, then

$$\operatorname{div}(T_1)(X) = 2\mathrm{d}f_1(X)(\sigma\varphi, \varphi) = 2\mathrm{d}f_1(X),$$

and so

$$\operatorname{div}\left(\frac{1}{4}T_1 - \frac{f_1}{2}\eta\right) = 0,\tag{3.12}$$

which is required by the Einstein equation in (1.9).
(ii) Similarly, if $D^2\varphi = f_2\varphi$ for some function $f_2 : Q^{n,r} \longrightarrow \mathbb{R}$ and φ is of constant length $|\varphi| = \pm 1$, then

$$\operatorname{div}(T_2)(X) = 2\mathrm{d}f_2(X)(\sigma\varphi, \varphi) = 2\mathrm{d}f_2(X),$$

and so

$$\operatorname{div}\left(\frac{1}{4}T_2 - \frac{f_2}{2}\eta\right) = 0.\tag{3.13}$$

From (3.12) and (3.13) we see that the Einstein equation

$$a\left\{\mathrm{Ric} - \frac{S}{2}\eta\right\} - \frac{c}{2}\eta = \frac{\epsilon}{4}T - \frac{\epsilon}{2}f\eta$$

of the CL-Einstein–Dirac equation (1.9) has a natural coupling structure. However, we should note that neither (3.12) nor (3.13) holds in general, unless (φ, φ) is of constant length.

We can rewrite the CL-Einstein–Dirac equation of type I

$$(\sqrt{-1})^r D\psi = f_1\psi,\tag{3.14}$$

$$a\left\{\mathrm{Ric} - \frac{S}{2}\eta\right\} - \frac{c}{2}\eta = \frac{\epsilon}{4}T_1 - \frac{\epsilon}{2}f_1\eta,\tag{3.15}$$

where

$$T_1(X, Y) = \left((\sqrt{-1})^r\{X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi\}, \psi\right),\tag{3.16}$$

in an equivalent form. Since contracting both sides of (3.15) gives

$$\epsilon(n-1)f_1 = a(n-2)S + cn,\tag{3.17}$$

one checks that the system (3.14) and (3.15) is actually equivalent to the system

$$\epsilon(\sqrt{-1})^r D\psi = \left\{\frac{a(n-2)}{n-1}S + \frac{cn}{n-1}\right\}\psi \tag{3.18}$$

and

$$a\left\{\mathrm{Ric} - \frac{S}{2(n-1)}\eta\right\} + \frac{c}{2(n-1)}\eta = \frac{\epsilon}{4}T_1. \tag{3.19}$$

Since the system (3.18) and (3.19) is similar to the classical Einstein–Dirac equation of type I, we are led to an analogue of the WK-equation in Definition 2.2.

Definition 3.2. A non-trivial spinor field ψ on $(Q^{n,r}, \eta), n \geq 3$, is called a *WW-spinor* if ψ satisfies the differential equation

$$\nabla_X\psi = (\sqrt{-1})^{3r}\left(-\frac{2a}{\epsilon}\right)\left\{\mathrm{Ric}(X) - \frac{S}{2(n-1)}X + \frac{c}{2a(n-1)}X\right\}\cdot\psi \tag{3.20}$$

for some constants $\epsilon, a, c \in \mathbb{R}, \epsilon \neq 0, a \neq 0$, and for all vector fields X.

Note that if the scalar curvature S of $(Q^{n,r}, \eta)$ is constant, then the WW-equation (3.20) is equivalent to the WK-equation (2.17). Because of (2.10), the length $|\psi|$ of any WW-spinor ψ is constant. It follows that, by rescaling the length $|\psi|$ if necessary, one may assume without loss of generality that any WW-spinor ψ is of unit length $|\psi| = \pm 1$ or of zero length $|\psi| = 0$. As any WK-spinor of positive (resp. negative) length is an Einstein spinor of type I, one then checks that any WW-spinor ψ of unit length is a CL-Einstein spinor of type I.

4. Constructing solutions of the CL-Einstein–Dirac equation of type II

Let η_1 and η_2, $\eta_2 = e^u\eta_1$, be conformally equivalent metrics on $Q^{n,r}$. By (2.3) there are natural isomorphisms $j : T(Q) \longrightarrow T(Q)$ and $j : \Sigma(Q)_{\eta_1} \longrightarrow \Sigma(Q)_{\eta_2}$ preserving the inner products of vectors and spinors as well as the Clifford multiplication:

$$\eta_2(jX, jY) = \eta_1(X, Y), \qquad \langle j\varphi_1, j\varphi_2\rangle_{\eta_2} = \langle\varphi_1, \varphi_2\rangle_{\eta_1},$$
$$(jX)\cdot(j\varphi) = j(X\cdot\varphi), \qquad X, Y \in \Gamma(T(Q)), \quad \varphi, \varphi_1, \varphi_2 \in \Gamma(\Sigma(Q)_{\eta_1}).$$

Denote by $\overline{X} := j(X)$ and $\overline{\varphi} := j(\varphi)$ the corresponding vector fields and spinor fields on $(Q^{n,r}, \eta_2)$, respectively. Then, for any spinor field ψ on $(Q^{n,r}, \eta_1)$, we have

$$\nabla^{\eta_2}_{\overline{X}}\overline{\psi} = e^{-\frac{u}{2}}\overline{\nabla^{\eta_1}_X\psi} - \frac{1}{4}\eta_2(\overline{X}, \mathrm{grad}_{\eta_2}(u))\overline{\psi} - \frac{1}{4}\overline{X}\cdot\mathrm{grad}_{\eta_2}(u)\cdot\overline{\psi}, \tag{4.1}$$

$$D_{\eta_2}\overline{\psi} = e^{-\frac{u}{2}}\overline{D_{\eta_1}\psi} + \frac{n-1}{4}\mathrm{grad}_{\eta_2}(u)\cdot\overline{\psi}, \tag{4.2}$$

$$\begin{aligned}(D_{\eta_2}\circ D_{\eta_2})\overline{\psi} \;=\; & e^{-u}\overline{(D_{\eta_1}\circ D_{\eta_1})\psi} - \frac{1}{2}e^{-\frac{u}{2}}\mathrm{grad}_{\eta_2}(u)\cdot\overline{D_{\eta_1}\psi} \\ & - \frac{n-1}{2}e^{-u}\overline{\nabla_{\mathrm{grad}_{\eta_1}(u)}\psi} + \frac{(n-1)^2}{16}|du|^2_{\eta_2}\overline{\psi} + \frac{n-1}{4}\triangle_{\eta_2}(u)\overline{\psi}.\end{aligned} \tag{4.3}$$

Now consider a special class of spinors.

Definition 4.1. A non-trivial spinor field ψ on $(Q^{n,r}, \eta), n \geq 3$, is called a *weakly T-parallel spinor* with conformal factor u if it is of constant length $|\psi| = \pm 1$ and the equation

$$\nabla_X \psi = -\frac{1}{4}\mathrm{d}u(X)\psi - \frac{1}{4}\beta(X) \cdot \mathrm{grad}(u) \cdot \psi \tag{4.4}$$

holds for all vector fields X, for a symmetric (1, 1)-tensor field β with

$$\mathrm{Tr}(\beta) = n,$$

and for a real-valued function $u : Q^{n,r} \longrightarrow \mathbb{R}$ such that $|\mathrm{d}u|$ has no zeros on an open dense subset of $Q^{n,r}$.

Note that if ψ is a parallel spinor on $(Q^{n,r}, \eta_1)$, then the pullback $\overline{\psi}$ of ψ is a weakly T-parallel spinor on $(Q^{n,r}, \eta_2)$ with β= the identity map. In the following, we identify via the metric η any exact 1-form "du" with the vector field "grad(u)" and (1, 1)-tensor field β with the induced (0, 2)-tensor field $\beta(X, Y) = \eta(X, \beta(Y))$.

Proposition 4.1. *Let $(Q^{n,r}, \eta)$ admit a weakly T-parallel spinor ψ solving Eq. (4.4). Then we have*

(i) $\beta(\mathrm{d}u) = \mathrm{d}u$,
(ii) $\nabla_{\mathrm{d}u}\psi = 0$,
(iii) $D\psi = \frac{n-1}{4}\mathrm{d}u \cdot \psi$,
(iv) $D^2\psi = \{\frac{(n-1)^2}{16}|\mathrm{d}u|^2 + \frac{n-1}{4}\triangle u\}\psi$, *where* $\triangle := -\mathrm{div} \circ \mathrm{grad}$,
(v) $S = \frac{1}{4}\{(n-1)^2 + 1 - |\beta|^2\}|\mathrm{d}u|^2 + (n-1)\triangle u$.

Proof. Since $(\sigma\psi, \psi) = 1$ is constant and β is symmetric,

$$0 = \sigma(\nabla_X\psi, \psi) = -\frac{1}{4}\mathrm{d}u(X) + \frac{1}{4}\eta(\beta(X), \mathrm{grad}(u)) = -\frac{1}{4}\mathrm{d}u(X) + \frac{1}{4}\eta(X, \beta(\mathrm{d}u)),$$

which proves part (i). Using (ii) and (iii), we compute

$$\begin{aligned} D^2\psi &= \frac{n-1}{4}D(\mathrm{d}u \cdot \psi) = \frac{n-1}{4}\triangle(u)\psi - \frac{n-1}{2}\nabla_{\mathrm{d}u}\psi - \frac{n-1}{4}\mathrm{d}u \cdot D\psi \\ &= \left\{\frac{n-1}{4}\triangle u + \frac{(n-1)^2}{16}|\mathrm{d}u|^2\right\}\psi, \end{aligned}$$

which proves part (iv). Substituting (iv) and (4.4) into the Schrödinger–Lichnerowicz formula $D^2\psi = \triangle\psi + \frac{S}{4}\psi$, one proves part (v). □

Remark 4.1. It is remarkable that when $Q^{n,r}$ is a closed manifold, the function $f_2 = \frac{(n-1)^2}{16}|\mathrm{d}u|^2 + \frac{n-1}{4}\triangle u$ in part (iv) of Proposition 4.1 cannot be constant. Suppose f_2 is a constant and hence an eigenvalue of D^2. Then f_2 must be equal to a "positive" constant λ^2 and for metric $\eta_1 := \mathrm{e}^{-u}\eta$, we have $\triangle_{\eta_1}(u) = \frac{n-3}{4}|\mathrm{d}u|^2_{\eta_1} + \frac{4}{n-1}\lambda^2\mathrm{e}^u$. The last relation is however a contradiction, since the left-hand side becomes zero after integration.

Let ψ be a weakly T-parallel spinor on $(Q^{n,r}, \eta)$ solving Eq. (4.4). Then, a direct computation gives

$$
\begin{aligned}
\frac{\epsilon}{4}T_2(X,Y) &= \frac{\epsilon\sigma}{4}\,(X\cdot\nabla_Y(D\psi)+Y\cdot\nabla_X(D\psi),\psi)\\
&\quad+\frac{\epsilon\sigma}{4}(-1)^r\,(X\cdot\nabla_Y\psi+Y\cdot\nabla_X\psi,D\psi)\\
&= \frac{\epsilon\sigma(n-1)}{16}\,(X\cdot\nabla_Y(\mathrm{d}u\cdot\psi)+Y\cdot\nabla_X(\mathrm{d}u\cdot\psi),\psi)\\
&\quad+\frac{\epsilon\sigma(n-1)}{16}(-1)^r\,(X\cdot\nabla_Y\psi+Y\cdot\nabla_X\psi,\mathrm{d}u\cdot\psi)\\
&= \frac{\epsilon\sigma(n-1)}{16}\,(X\cdot\nabla_Y\mathrm{d}u\cdot\psi+Y\cdot\nabla_X\mathrm{d}u\cdot\psi,\psi)\\
&\quad-\frac{\epsilon\sigma(n-1)}{64}\,(X\cdot\mathrm{d}u\cdot\{\mathrm{d}u(Y)\psi+\beta(Y)\cdot\mathrm{d}u\cdot\psi\}\\
&\quad+Y\cdot\mathrm{d}u\cdot\{\mathrm{d}u(X)\psi+\beta(X)\cdot\mathrm{d}u\cdot\psi\},\psi)\\
&\quad-\frac{\epsilon\sigma(n-1)}{64}(-1)^r\,(X\cdot\{\mathrm{d}u(Y)\psi+\beta(Y)\cdot\mathrm{d}u\cdot\psi\}\\
&\quad+Y\cdot\{\mathrm{d}u(X)\psi+\beta(X)\cdot\mathrm{d}u\cdot\psi\},\mathrm{d}u\cdot\psi)\\
&= -\frac{\epsilon(n-1)}{8}\eta(X,\nabla_Y\mathrm{d}u)-\frac{\epsilon(n-1)}{16}\mathrm{d}u(X)\mathrm{d}u(Y)+\frac{\epsilon(n-1)}{16}|\mathrm{d}u|^2\beta(X,Y).
\end{aligned}
$$

Guided by the last computation, one immediately proves:

Theorem 4.1. *Let ψ be a weakly T-parallel spinor on $(Q^{n,r},\eta)$ such that β and u are related to the Ricci tensor and the scalar curvature of $(Q^{n,r},\eta)$ by*

$$
\begin{aligned}
|\mathrm{d}u|^2\beta(X,Y) &= \frac{4}{n-2}\left\{\mathrm{Ric}(X,Y)-\frac{1}{2}S\eta(X,Y)\right\}-\frac{2c}{a(n-2)}\eta(X,Y)\\
&\quad+2\eta(X,\nabla_Y(\mathrm{d}u))+\mathrm{d}u(X)\mathrm{d}u(Y)\\
&\quad+\left\{\frac{n-1}{2}|\mathrm{d}u|^2+2\triangle u\right\}\eta(X,Y),
\end{aligned}
\tag{4.5}
$$

where $a,c\in\mathbb{R}$, $a\neq 0$, are real numbers. Then ψ becomes a solution of the CL-Einstein–Dirac equation of type II (i.e., the system (1.9) and (1.11)), where the characteristic function f is given by

$$
f=\frac{(n-1)^2}{16}|\mathrm{d}u|^2+\frac{n-1}{4}\triangle u
$$

and the parameter ϵ should be chosen to satisfy

$$
\epsilon=\frac{4a(n-2)}{n-1}.
$$

Definition 4.2. A non-trivial spinor field ψ on $(Q^{n,r},\eta), n\geq 3$, is called a *weakly parallel spinor* (briefly, WP-spinor) with conformal factor u if it is a weakly T-parallel spinor with conformal factor u and satisfies (4.5) for some constants $a,c\in\mathbb{R}$, $a\neq 0$.

Definition 4.3. A non-trivial spinor field ψ on $(Q^{n,r},\eta), n\geq 3$, is called a *reduced weakly parallel spinor* (briefly, reduced WP-spinor) with conformal factor u if it is of constant length $|\psi|=\pm 1$ and the differential equation

$$|\mathrm{d}u|^2\nabla_X\psi = -\frac{1}{n-2}\left\{\mathrm{Ric}(X) - \frac{S}{n}X\right\}\cdot \mathrm{d}u\cdot\psi \tag{4.6}$$

holds for all vector fields X and for a real-valued function $u : Q^{n,r} \longrightarrow \mathbb{R}$ with such properties that $|\mathrm{d}u|$ has no zeros on an open dense subset of $Q^{n,r}$ and e^u is proportional to the scalar curvature S, i.e.,

$$S = c^*\mathrm{e}^u, \qquad c^* \in \mathbb{R}. \tag{4.7}$$

Note that (4.6) generalizes the equation $\nabla_X\psi = 0$ for parallel spinors and that any reduced WP-spinor ψ is a harmonic spinor $D\psi = 0$. Applying (4.6) to $0 = \sigma\cdot|\mathrm{d}u|^2(\nabla_X\psi, \psi)$, one shows:

Proposition 4.2. *Let $(Q^{n,r}, \eta)$ admit a reduced WP-spinor ψ with conformal factor u. Then*

$$\nabla_{\mathrm{d}u}\psi = 0 \qquad \textit{and} \qquad \mathrm{Ric}(\mathrm{d}u) = \frac{S}{n}\mathrm{d}u.$$

We are going to prove that Eq. (4.5) for WP-spinors is conformally equivalent to Eq. (4.6) for reduced WP-spinors. Consider conformally equivalent metrics $\eta_2 = \mathrm{e}^u\eta_1$ on $Q^{n,r}$. Let $(F_1, \dots, F_n)$ be a local η_1-orthonormal frame field on $Q^{n,r}$. Then $(\overline{F}_1 := \mathrm{e}^{-\frac{u}{2}}F_1, \dots, \overline{F}_n := \mathrm{e}^{-\frac{u}{2}}F_n)$ is η_2-orthonormal. Since the Ricci tensors Ric_{η_2} and Ric_{η_1} are related by

$$\begin{aligned}
&\mathrm{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_j) - \mathrm{e}^{-u}\mathrm{Ric}_{\eta_1}(F_i, F_j)\\
&\quad = -\frac{n-2}{2}\eta_2(\overline{F}_i, \nabla^{\eta_2}_{\overline{F}_j}(\mathrm{grad}_{\eta_2}u)) - \frac{n-2}{4}\mathrm{d}u(\overline{F}_i)\mathrm{d}u(\overline{F}_j)\\
&\qquad + \frac{1}{2}\triangle_{\eta_2}(u)\eta_2(\overline{F}_i, \overline{F}_j) + \frac{n-2}{4}|\mathrm{d}u|^2_{\eta_2}\eta_2(\overline{F}_i, \overline{F}_j)
\end{aligned}$$

and the scalar curvatures S_{η_2} and S_{η_1} by

$$S_{\eta_2} - \mathrm{e}^{-u}S_{\eta_1} = (n-1)\triangle_{\eta_2}(u) + \frac{(n-1)(n-2)}{4}|\mathrm{d}u|^2_{\eta_2},$$

we have in particular the following formula.

Lemma 4.1.

$$\begin{aligned}
&\mathrm{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_j) - \frac{1}{2}S_{\eta_2}\eta_2(\overline{F}_i, \overline{F}_j)\\
&\quad = \mathrm{e}^{-u}\left\{\mathrm{Ric}_{\eta_1}(F_i, F_j) - \frac{1}{2}S_{\eta_1}\eta_1(F_i, F_j)\right\}\\
&\qquad - \frac{n-2}{2}\eta_2(\overline{F}_i, \nabla^{\eta_2}_{\overline{F}_j}(\mathrm{grad}_{\eta_2}u)) - \frac{n-2}{4}\mathrm{d}u(\overline{F}_i)\mathrm{d}u(\overline{F}_j)\\
&\qquad - \frac{n-2}{2}\triangle_{\eta_2}(u)\eta_2(\overline{F}_i, \overline{F}_j) - \frac{(n-2)(n-3)}{8}|\mathrm{d}u|^2_{\eta_2}\eta_2(\overline{F}_i, \overline{F}_j).
\end{aligned}$$

Theorem 4.2. *A non-trivial spinor field ψ on $(Q^{n,r}, \eta_1)$ is a reduced WP-spinor with conformal factor u if and only if the pullback $\overline{\psi}$ of ψ is a WP-spinor on $(Q^{n,r}, \eta_2 = \mathrm{e}^u\eta_1)$ with conformal factor u.*

Proof. We first prove the necessity. Let ψ be a reduced WP-spinor on $(Q^{n,r}, \eta_1)$ with conformal factor u. In the notation of (4.1), we have

$$|\mathrm{d}u|^2_{\eta_2} \nabla^{\eta_2}_{\overline{X}} \overline{\psi}$$
$$= -\frac{1}{n-2} \mathrm{e}^{-u} \left\{ \overline{\mathrm{Ric}_{\eta_1}(X)} - \frac{1}{n} S_{\eta_1} \overline{X} \right\} \cdot \mathrm{grad}_{\eta_2}(u) \cdot \overline{\psi}$$
$$- \frac{1}{4} |\mathrm{d}u|^2_{\eta_2} \eta_2(\overline{X}, \mathrm{grad}_{\eta_2}(u)) \overline{\psi} - \frac{1}{4} |\mathrm{d}u\,|^2_{\eta_2}\, \overline{X} \cdot \mathrm{grad}_{\eta_2}(u) \cdot \overline{\psi}$$

and hence

$$\nabla^{\eta_2}_{\overline{X}} \overline{\psi} = -\frac{1}{4} \eta_2(\overline{X}, \mathrm{grad}_{\eta_2}(u)) \overline{\psi} - \frac{1}{4} \gamma(\overline{X}) \cdot \mathrm{grad}_{\eta_2}(u) \cdot \overline{\psi}, \tag{4.8}$$

where γ is a symmetric tensor field defined by

$$|\mathrm{d}u|^2_{\eta_2} \gamma(\overline{X}, \overline{Y})$$
$$= \frac{4}{n-2} \mathrm{e}^{-u} \left\{ \mathrm{Ric}_{\eta_1}(X, Y) - \frac{1}{n} S_{\eta_1} \eta_1(X, Y) \right\} + |\mathrm{d}u|^2_{\eta_2} \eta_1(X, Y). \tag{4.9}$$

On the other hand, using Lemma 4.1, we compute

$$\begin{aligned}\Phi(\overline{X}, \overline{Y}) :=\ & \frac{4}{n-2} \left\{ \mathrm{Ric}_{\eta_2}(\overline{X}, \overline{Y}) - \frac{1}{2} S_{\eta_2} \eta_2(\overline{X}, \overline{Y}) \right\} - \frac{2c}{a(n-2)} \eta_2(\overline{X}, \overline{Y}) \\ & + 2\eta_2(\overline{X}, \nabla^{\eta_2}_{\overline{Y}}(\mathrm{grad}_{\eta_2}(u))) + \mathrm{d}u(\overline{X}) \mathrm{d}u(\overline{Y}) \\ & + \left\{ \frac{n-1}{2} |\mathrm{d}u|^2_{\eta_2} + 2 \triangle_{\eta_2}(u) \right\} \eta_2(\overline{X}, \overline{Y}) \\ =\ & \frac{4\mathrm{e}^{-u}}{n-2} \left\{ \mathrm{Ric}_{\eta_1}(X, Y) - \frac{1}{2} S_{\eta_1} \eta_1(X, Y) - \frac{c\mathrm{e}^u}{2a} \eta_1(X, Y) \right\} + |\mathrm{d}u|^2_{\eta_2} \eta_1(X, Y).\end{aligned}$$

Choose the parameters $a, c \in \mathbb{R}$ such that the constant c^* in (4.7) satisfies

$$c^* = -\frac{cn}{a(n-2)}.$$

Then $S_{\eta_1} = -\frac{cn}{a(n-2)} \mathrm{e}^u$ and

$$\Phi(\overline{X}, \overline{Y}) = |\mathrm{d}u|^2_{\eta_2} \gamma(\overline{X}, \overline{Y}). \tag{4.10}$$

From (4.8)–(4.10), we conclude that $\overline{\psi}$ is a weakly T-parallel spinor on $(Q^{n,r}, \eta_2 = \mathrm{e}^u \eta_1)$ satisfying (4.5), i.e., $\overline{\psi}$ is a WP-spinor. In order to prove the sufficiency, we reverse the process of the proof for the necessity. Let $\overline{\psi}$ be a WP-spinor on $(Q^{n,r}, \eta_2 = \mathrm{e}^u \eta_1)$. Then we have

$$|\mathrm{d}u|^2_{\eta_2} \beta(\overline{X}, \overline{Y})$$
$$= \frac{4\mathrm{e}^{-u}}{n-2} \left\{ \mathrm{Ric}_{\eta_1}(X, Y) - \frac{1}{2} S_{\eta_1} \eta_1(X, Y) - \frac{c\mathrm{e}^u}{2a} \eta_1(X, Y) \right\} + |\mathrm{d}u|^2_{\eta_2} \eta_1(X, Y).$$

Contracting both sides of this equation gives

$$S_{\eta_1} = -\frac{cn}{a(n-2)} \mathrm{e}^u.$$

Using (4.1), one verifies that ψ satisfies Eq. (4.6) indeed. □

5. An existence theorem for WK-spinors and that for reduced WP-spinors

We show that every parallel spinor may evolve to a WK-spinor (resp. a reduced WP-spinor). We give a description for the evolution in a more general way than that given in Section 5 of [7].

Let (M^n, g_M) be a Riemannian manifold, and let $(\mathbb{R}, g_{\mathbb{R}})$ be the real line with the standard metric. Let $(Q^{n+1} = M^n \times \mathbb{R}, \eta_1 = g_M + \chi(n+1)g_{\mathbb{R}})$, $\chi(n+1) = \pm 1$, be the pseudo-Riemannian product manifold. We will write $g_{\mathbb{R}} = \mathrm{d}t \otimes \mathrm{d}t$ using the standard coordinate $t \in \mathbb{R}$ and regard η_1 as a reference metric on Q^{n+1}. Let $(F_1, \ldots, F_n)$ denote a local η_1-orthonormal frame field on (M^n, g_M) as well as its lift to (Q^{n+1}, η_1). Let $F_{n+1} = \frac{\mathrm{d}}{\mathrm{d}t}$ denote the unit vector field on $(\mathbb{R}, g_{\mathbb{R}})$ as well as the lift to (Q^{n+1}, η_1). We consider a doubly warped product of g_M and $g_{\mathbb{R}}$:

$$\eta_2 = A^2 \left(\sum_{i=1}^{n} F^i \otimes F^i \right) + \chi(n+1) B^2 \mathrm{d}t \otimes \mathrm{d}t, \tag{5.1}$$

where $A = A(t)$, $B = B(t) : \mathbb{R} \longrightarrow \mathbb{R}$ are positive functions on $\mathbb{R}$ and $\{F^i = \eta_1(F_i, \cdot)\}$ is the dual frame field of $\{F_i\}$. Let g_{M_t} be the metric on slice $M_t := M^n \times \{t\}$, $t \in \mathbb{R}$, of the foliation $(Q^{n+1} = M^n \times \mathbb{R}, \eta_1)$ induced by the reference metric η_1, and let $\nabla^{g_{M_t}}$ be the Levi-Civita connection. Then the Levi-Civita connection ∇^{η_2} of (Q^{n+1}, η_2) is related to $\nabla^{g_{M_t}}$ by

$$\nabla^{\eta_2}_{\overline{F}_i} \overline{F}_j = A^{-2} \nabla^{g_{M_t}}_{F_i} F_j - \chi(n+1)\delta_{ij} B^{-2} A^{-1} A_t F_{n+1}, \tag{5.2}$$

$$\nabla^{\eta_2}_{\overline{F}_{n+1}} \overline{F}_j = \nabla^{\eta_2}_{\overline{F}_{n+1}} \overline{F}_{n+1} = 0, \qquad 1 \le i, j \le n, \tag{5.3}$$

where $(\overline{F}_1 := A^{-1}F_1, \ldots, \overline{F}_n := A^{-1}F_n, \overline{F}_{n+1} := B^{-1}F_{n+1})$ is a η_2-orthonormal frame field and A_t indicates the derivative $A_t = \mathrm{d}A(F_{n+1})$. The second fundamental form $\Theta_{\eta_2} = -\nabla^{\eta_2}\overline{F}_{n+1}$ of slice M_t is expressed as

$$\Theta_{\eta_2}(\overline{F}_j) = -B^{-1}A^{-1}A_t \overline{F}_j, \qquad 1 \le j \le n. \tag{5.4}$$

Furthermore, the Ricci tensor Ric_{η_2} and the scalar curvature S_{η_2} of (Q^{n+1}, η_2) are related to the Ricci tensor $\mathrm{Ric}_{g_{M_t}}$ and the scalar curvature S_{M_t} of slice (M_t, g_{M_t}) by

$$\begin{aligned}\mathrm{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_j) \;=\; & A^{-2}\mathrm{Ric}_{g_{M_t}}(F_i, F_j) - \chi(n+1)(n-1)B^{-2}A^{-2}A_t A_t \delta_{ij} \\ & + \chi(n+1)\{B^{-3}A^{-1}B_t A_t - B^{-2}A^{-1}A_{tt}\}\delta_{ij},\end{aligned} \tag{5.5}$$

$$\mathrm{Ric}_{\eta_2}(\overline{F}_{n+1}, \overline{F}_{n+1}) = nB^{-2}A^{-1}(B^{-1}B_t A_t - A_{tt}), \tag{5.6}$$

$$\mathrm{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_{n+1}) = 0, \tag{5.7}$$

$$\begin{aligned}S_{\eta_2} \;=\; & A^{-2}S_{g_{M_t}} - \chi(n+1)n(n-1)B^{-2}A^{-2}A_t A_t \\ & + \chi(n+1)2n\{B^{-3}A^{-1}B_t A_t - B^{-2}A^{-1}A_{tt}\},\end{aligned} \tag{5.8}$$

where $A_{tt} = (A_t)_t$ indicates the second derivative. From now on, we are interested in a special case that the warping functions A and B are related by

$$B = (A^p)_t = pA^{p-1}A_t, \qquad p \neq 0 \in \mathbb{R}. \tag{5.9}$$

Definition 5.1. A doubly warped product (5.1) is called a *(Y)-warped product* of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with warping function A and (Y)-factor p if the relation (5.9) is satisfied for some constant $p \neq 0 \in \mathbb{R}$.

Proposition 5.1. *Let $(Q^{n+1} = M^n \times \mathbb{R}, \eta_2)$ be a (Y)-warped product of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with warping function A and (Y)-factor p. Then the formulas (5.4)–(5.8) simplify to*

(i) $\Theta_{\eta_2}(\overline{F}_i, \overline{F}_j) = -p^{-1}A^{-p}\delta_{ij}, 1 \leq i, j \leq n,$
(ii) $\mathrm{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_j) = A^{-2}\mathrm{Ric}_{g_{M_t}}(F_i, F_j) + \chi(n+1)(p-n)p^{-2}A^{-2p}\delta_{ij},$
(iii) $\mathrm{Ric}_{\eta_2}(\overline{F}_{n+1}, \overline{F}_{n+1}) = n(p-1)p^{-2}A^{-2p},$
(iv) $\mathrm{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_{n+1}) = 0,$
(v) $S_{\eta_2} = A^{-2}S_{g_{M_t}} + \chi(n+1)n(2p-n-1)p^{-2}A^{-2p}.$

An argument similar to that of Proposition 5.1 of [7] shows:

Proposition 5.2. *Let $(Q^{n+1} = M^n \times \mathbb{R}, \eta_2)$ be a (Y)-warped product of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with warping function A and (Y)-factor $\frac{n}{2}$. Assume that (M^n, g_M) is Ricci-flat. Then the weak Killing equation (2.17), in the case of $b = 0$, is equivalent to the system of differential equations*

$$\nabla^{g_{M_t}}_V \psi = 0 \qquad \text{and} \qquad \nabla^{\eta_2}_{\overline{F}_{n+1}} \psi = -(\sqrt{-1})^{3r}\nu_1 \overline{F}_{n+1} \cdot \psi + \frac{1}{2}\mathrm{Tr}_{g_{M_t}}(\Theta_{\eta_2})\psi,$$

where V is an arbitrary vector field on Q^{n+1} with $\eta_2(V, \overline{F}_{n+1}) = 0$.

Proposition 5.3. *Let $(Q^{n+1} = M^n \times \mathbb{R}, \eta_2)$ be a (Y)-warped product of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with warping function A and (Y)-factor $\frac{n+1}{2}$. Assume that (M^n, g_M) is Ricci-flat. Then the reduced WP-equation in Definition 4.3 (in the case that we set $u = -\log A$) is equivalent to the system of differential equations*

$$\nabla^{g_{M_t}}_V \psi = 0 \qquad \text{and} \qquad \nabla^{\eta_2}_{\overline{F}_{n+1}} \psi = \frac{1}{2}\mathrm{Tr}_{g_{M_t}}(\Theta_{\eta_2})\psi,$$

where V is an arbitrary vector field on Q^{n+1} with $\eta_2(V, \overline{F}_{n+1}) = 0$.

Proof. Since $u = -\log A$, we have

$$|\mathrm{d}u|^2_{\eta_2} = \chi(n+1)p^{-2}A^{-2p},$$
$$\mathrm{grad}_{\eta_2}(u) = -\chi(n+1)p^{-1}A^{-p}\overline{F}_{n+1}.$$

Moreover, by part (v) of Proposition 5.1, the scalar curvature $S_{\eta_2} = 0$ vanishes. Thus the reduced WP-equation becomes

$$\begin{aligned}\nabla^{\eta_2}_V \psi &= -\frac{1}{n-1}\mathrm{Ric}_{\eta_2}(V) \cdot \frac{\mathrm{grad}_{\eta_2}(u)}{|\mathrm{d}u|^2_{\eta_2}} \cdot \psi \\ &= \frac{p}{n-1}A^p\mathrm{Ric}_{\eta_2}(V) \cdot \overline{F}_{n+1} \cdot \psi \\ &= -\chi(n+1)\frac{1}{n+1}A^{-\frac{n+1}{2}} V \cdot \overline{F}_{n+1} \cdot \psi \qquad (5.10)\end{aligned}$$

and

$$\nabla^{\eta_2}_{\overline{F}_{n+1}}\psi = -\frac{1}{n-1}\mathrm{Ric}_{\eta_2}(\overline{F}_{n+1})\cdot\frac{\mathrm{grad}_{\eta_2}(u)}{|\mathrm{d}u|^2_{\eta_2}}\cdot\psi$$
$$= -\frac{n}{n+1}A^{-\frac{n+1}{2}}\psi = \frac{1}{2}\mathrm{Tr}_{g_{M_t}}(\Theta_{\eta_2})\psi. \tag{5.11}$$

On the other hand,

$$\nabla^{\eta_2}_{V}\psi = \nabla^{g_{M_t}}_{V}\psi + \chi(n+1)\frac{1}{2}\Theta_{\eta_2}(V)\cdot\overline{F}_{n+1}\cdot\psi$$
$$= \nabla^{g_{M_t}}_{V}\psi - \chi(n+1)\frac{1}{n+1}A^{-\frac{n+1}{2}}V\cdot\overline{F}_{n+1}\cdot\psi. \tag{5.12}$$

From (5.10)–(5.12) we conclude the proof. □

Following a standard argument in the proof of Proposition 5.2 and Theorem 5.1 of [7] in pseudo-Riemannian signature, we now establish the following existence theorems.

Theorem 5.1. *Let $(Q^{n+1} = M^n \times \mathbb{R}, \eta_2)$ be a (Y)-warped product of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with (Y)-factor $\frac{n}{2}$. If (M^n, g_M) admits a parallel spinor, then for any real number $\lambda_Q \neq 0$, (Q^{n+1}, η_2) admits a WK-spinor to WK-number $(\sqrt{-1})^{3r}\lambda_Q$, where $r = 0$ if $\chi(n+1) = 1$ and $r = 1$ if $\chi(n+1) = -1$, respectively.*

Theorem 5.2. *Let $(Q^{n+1} = M^n \times \mathbb{R}, \eta_2)$ be a (Y)-warped product of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with (Y)-factor $\frac{n+1}{2}$. If (M^n, g_M) admits a parallel spinor, then (Q^{n+1}, η_2) admits a reduced WP-spinor that is not a parallel spinor.*

Theorem 5.1 above improves Theorem 5.1 of [7], since (Y)-warped products of (M^n, g_M) and $(\mathbb{R}, g_{\mathbb{R}})$ with (Y)-factor $\frac{n}{2}$ essentially generalize the metrics in Lemma 5.3 of [7].

References

[1] H. Baum, Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten, Teubner-Verlag, Leipzig, 1981.
[2] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, Twistors and Killing Spinors on Riemannian Manifolds, Teubner, Leipzig/Stuttgart, 1991.
[3] D. Bleecker, Gauge Theory and Variational Principles, Addison-Wesley, Mass., 1981.
[4] J.P. Bourguignon, P. Gauduchon, Spineurs, Opérateurs de Dirac et Variations de Mé triques, Comm. Math. Phys. 144 (1992) 581–599.
[5] Th. Friedrich, Solutions of the Einstein–Dirac equation on Riemannian 3-manifolds with constant scalar curvature, J. Geom. Phys. 36 (2000) 199–210.
[6] Th. Friedrich, E.C. Kim, The Einstein–Dirac equation on Riemannian spin manifolds, J. Geom. Phys. 33 (2000) 128–172.
[7] E.C. Kim, A local existence theorem for the Einstein–Dirac equation, J. Geom. Phys. 44 (2002) 376–405.
[8] M. Wang, Parallel spinors and parallel forms, Ann. Global Anal. Geom. 7 (1989) 59–68.